ISRAEL JOURNAL OF MATHEMATICS, Vol. 63, No. 3, 1988

QUASILINEAR ELLIPTICITY AND THE DIRICHLET PROBLEM

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ABSTRACT

In this paper we prove some existence theorems for the Dirichlet problem in $\dot{W}^{1,2}(Q)$ for a quasilinear elliptic equation under assumptions of the Ambrosetti–Prodi type. We also discuss the solvability of this problem, in a nonresonant case, with boundary data in $L^2(\partial Q)$ which leads in a natural way to the Dirichlet problem in a weighted Sobolev space.

1. Introduction

In recent years the Dirichlet problem for semilinear elliptic equations with a nonlinear part crossing the first eigenvalue has been widely studied by many authors. The Dirichlet problem with nonlinearities interacting with the spectrum of the elliptic operator appears to have been first noted in the literature by Ambrosetti–Prodi [2]. Their result can be summarized as follows.

Let $Q \subset \mathbf{R}_n$ be an open bounded domain with a smooth boundary and let $g \in C^2(\mathbf{R})$ with $g''(u) > 0$ for all $u \in \mathbf{R}$. Suppose that

$$\lim_{u \to -\infty} \frac{g(u)}{u} = \alpha \quad \text{and} \quad \lim_{u \to \infty} \frac{g(u)}{u} = \beta$$

exist and that $0 < \alpha < \lambda_1 < \beta < \lambda_2$, where λ_1 and λ_2 are the first and the second eigenvalue of the Laplace operator. Under these assumptions they showed that the Hölder space $C^\alpha(\bar{Q})$, $0 < \alpha < 1$, admits a decomposition $C^\alpha(\bar{Q}) = E_0 \cup E_1 \cup E_2$, where E_0 and E_2 are disjoint open sets with $E_1 = \partial E_0 = \partial E_2$ such that the Dirichlet problem

Received December 29, 1987 and in revised form June 10, 1988

$$\Delta u + g(u) = h(x) \quad \text{in } Q,$$

$$u(x) = 0 \quad \text{on } \partial Q,$$

has exactly j solutions for $h \in E_j$, $j = 0, 1, 2$. A description of the sets E_j was given by Berger and Podolak [5]. For further extensions for more general elliptic linear operators we refer to Amann and Hess [1], Kazdan and Warner [18] (see also survey articles: de Figueiredo [16], Lazar and McKenna [19]). The investigation of the existence of multiple solutions in a Sobolev space $\dot{W}^{1,2}(Q)$, in the case where the Laplace operator is replaced by a more general elliptic operator with measurable coefficients, has been initiated by McKenna and Walter [20].

The purpose of this article is to study the existence of solutions in $\dot{W}^{1,2}(Q)$ of the Dirichlet problem for a quasilinear elliptic operator of the form

$$Lu = -\sum_{i,j=1}^{n} D_i(a_{ij}(x, u)D_j u) = f(u) + h(x).$$

We associate with each $v \in L^2(Q)$ the elliptic operator

$$L^v = -\sum_{i,j=1}^{n} D_i(a_{ij}(x, v)D_j \cdot)$$

and by $\lambda_1(v)$ we denote the first eigenvalue of the operator L^v. Let

$$\kappa_1 = \inf\{\lambda_1(v); v \in L^2(Q)\} \quad \text{and} \quad \kappa_2 = \sup\{\lambda_1(v); v \in L^2(Q)\}.$$

Then under the assumption of the Ambrosetti–Prodi type $f'(-\infty) < \kappa_1 \leqq \kappa_2 \leqq f'(\infty)$ we establish some existence and non-existence results for the Dirichlet problem in $\dot{W}^{1,2}(Q)$ (see Sections 2, 3 and 4). To obtain the existence of at least two solutions we use the degree theory for pseudomonotone operators (see Berkovits–Mustonen [6], Berkovits [7] and Browder [8]) and the concept of the G-covergence of elliptic operators (Spagnolo [24]). In Section 5 we assume that all coefficients have uniform limits as $|u| \to \infty$. Consequently we can associate with these limit coefficients an elliptic operator. The first eigenvalue μ_1 of the limit operator always satisfies the inequality $\kappa_1 \leqq \mu_1$. Some existence results, in the case $\mu_1 = \kappa_1$, were obtained [13]. In Section 5 we consider the case $\kappa_1 < \mu_1$. We establish some existence theorems in the case where the nonlinearity is not in resonance with the limit operator, that is, the values of f do not interact with μ_1. We also discuss the existence results when coefficients of L have only one sided limits at infinity. The existence results in the case where f interacts with the first eigenvalue of the limit operator are given in Section 6. In particular we obtain a more general version of a theorem of the Ambrosetti–Prodi type obtained in Section 4. In the final Section 7 we briefly discuss the Dirichlet problem in resonance and compare our results with the recent work of Shapiro [23]. In Sections 5 and 6 we assume that the boundary data belong to $L^2(\partial Q)$. We point out here that this assumption leads in a natural way to the Dirichlet problem in a weighted Sobolev space $\tilde{W}^{1,2}(Q)$.

2. Preliminaries

Let $Q \subset \mathbf{R}_n$ be a bounded domain. In Q we consider the Dirichlet problem

$$(1_t) \qquad Lu = -\sum_{i,j=1}^{n} D_i(a_{ij}(x,u)D_j u) = f(u) + t\Phi(x) + h(x) \qquad \text{in } Q,$$

$$(2) \qquad u(x) = 0 \qquad \text{on } \partial Q,$$

where t is a real parameter, $h \in L^\infty(Q)$ and $\Phi \in C(\bar{Q})$, with $\Phi(x) > 0$ on $\bar{Q}$, are given functions.

We assume the following throughout the paper;

(A) There exist constants $\alpha > 0$ and $\beta > 0$ such that

$$\alpha|\xi|^2 \leqq \sum_{i,j=1}^{n} a_{ij}(x,u)\xi_i\xi_j \leqq \beta|\xi|^2$$

for all $\xi \in \mathbf{R}_n$ and $(x,u) \in Q \times \mathbf{R}$, $a_{ij} = a_{ji}$ $(i,j = 1,\dots,n)$. Moreover the functions $a_{ij}(x,u)$ $(i,j = 1,\dots,n)$ satisfy the Carathéodory conditions, that is, for each $u \in \mathbf{R}$ the functions $a_{ij}(\cdot,u)$ are measurable on Q and for a.e. $x \in Q$ the functions $a_{ij}(x,\cdot)$ are continuous on $\mathbf{R}$.

To formulate the assumption on the nonlinearity f we need some terminology.

We associate with every $v \in L^2(Q)$ a linear operator of the elliptic type given by

$$L^v = -\sum_{i,j=1}^{n} D_i(a_{ij}(x,v)D_j\,\cdot).$$

Let $\lambda_1(v)$ be the first eigenvalue of L^v and let us define

$$\kappa_1 = \inf\{\lambda_1(v);\, v \in L^2(Q)\} \quad \text{and} \quad \kappa_2 = \sup\{\lambda_1(v);\, v \in L^2(Q)\}.$$

It is known that $\lambda_1(v)$ is simple and the corresponding eigenfunctions φ_v can be taken positive on Q and normalized by

$$\int_Q \varphi_v(x)^2 dx = 1.$$

Using the variational characterization of the first eigenvalue it is easy to see that $\kappa_1 > 0$. It is obvious that $\kappa_2 < \infty$.

We assume that the nonlinearity $f \in C^1(\mathbf{R})$ and satisfies the Ambrosetti–Prodi type condition

$$\text{(B)} \qquad -\infty < f'(-\infty) < \kappa_1 \leqq \kappa_2 < f'(\infty) < \infty.$$

Finally, we recall that a function $u \in \dot{W}^{1,2}(Q)$ is a solution of (1_t), (2) if

$$\text{(3)} \qquad \int_Q \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j v \, dx = \int_Q [f(u)v + t\Phi(x)v + h(x)v] dx$$

for every $v \in W^{1,2}(Q)$ with a compact support in Q.

We commence with the following lemma.

Lemma 1. *Let φ_v be the first eigenfunction corresponding to the operator L^v, $v \in L^2(Q)$. Then*

$$\inf_{v \in L^2(Q)} \int_Q \varphi_v(x) dx > 0.$$

Proof. Let $\{\varphi_{v_m}\}$ be a minimizing sequence, that is

$$\lim_{m \to \infty} \int_Q \varphi_{v_m}(x) dx = \inf_{v \in L^2(Q)} \int_Q \varphi_v(x) dx.$$

For each m we have

$$\int_Q \sum_{i,j=1}^n a_{ij}(x, v_m) D_i \varphi_{v_m} D_j \varphi_{v_m} dx = \lambda_1(v_m) \int_Q \varphi_{v_m}^2 dx,$$

and hence, by the ellipticity condition, the sequence φ_{v_m} is also bounded in $W^{1,2}(Q)$. Consequently, we may assume that the sequence φ_{v_m} converges to a function $\varphi \in W^{1,2}(Q)$ weakly in $W^{1,2}(Q)$ and strongly in $L^2(Q)$ (Theorem 7.22 in [17]). On the other hand $\int_Q \varphi_{v_m}^2 dx = 1$ for each m, therefore $\int_Q \varphi^2 dx = 1$, $\varphi \geqq 0$ and $\varphi \not\equiv 0$ on Q. We also have

$$0<\int_Q \varphi(x)dx=\inf_{v\in L^2(Q)}\int_Q \varphi_v(x)dx$$

and this completes the proof.

To proceed further we observe that the assumption (B) implies the existence of constants $C>0$ and $0<\delta_1<\kappa_1\leqq\kappa_2<\delta_2$ such that

$$f(s)\geqq\delta_1 s-C \qquad \text{for all } s\in\mathbf{R} \tag{4}$$

and

$$f(s)\geqq\delta_2 s-C \qquad \text{for all } s\in\mathbf{R}. \tag{5}$$

LEMMA 2. *There exists a constant t_0 such that for $t>t_0$ the problem (1_t), (2) has no solution.*

PROOF. Let $u\in\mathring{W}^{1,2}(Q)$ be a solution of the problem (1_t), (2) and let φ_u be the first eigenfunction for the operator

$$L^u=-\sum_{i,j=1}^n D_i(a_{ij}(x,u)D_j\,\cdot).$$

Taking φ_u as a test function in (3) we obtain

$$\int_Q\sum_{i,j=1}^n a_{ij}(x,u)D_iuD_j\varphi_u dx=\int_Q[f(u)\varphi_u+t\Phi(x)\varphi_u+h(x)\varphi_u]dx.$$

Using the estimates (4) and (5) we get

$$\int_Q\lambda_1(u)u\varphi_u dx\geqq\int_Q(\delta_1u-C)\varphi_u dx+t\int_Q\Phi\varphi_u dx+\int_Q h\varphi_u dx$$

and

$$\int_Q\lambda_1(u)\varphi_u dx\geqq\int_Q(\delta_2u-C)\varphi_u dx+t\int_Q\Phi\varphi_u dx+\int_Q h\varphi_u dx.$$

The last two inequalities yield

$$t\int_Q\varphi_u\Phi\,dx\leqq\int_Q(\lambda_1(u)-\delta_1)u\varphi_u dx+C\int_Q\varphi_u dx-\int_Q h\varphi_u dx \tag{6}$$

and

$$t\int_Q\varphi_u\Phi\,dx\leqq\int_Q(\lambda_1(u)-\delta_2)u\varphi_u dx+C\int_Q\varphi_u dx-\int_Q h\varphi_u dx. \tag{7}$$

We notice that $\delta_1 < \lambda_1(u) < \delta_2$, hence it follows from (6) that

$$t\int_Q \Phi\varphi_u dx \leqq C\int_Q \varphi_u dx - \int_Q h\varphi_u dx \tag{8}$$

provided $\int_Q u\varphi_u dx \leqq 0$. Similarly, if $\int_Q u\varphi_u dx \geqq 0$ then again (7) implies the inequality (8). Since we always assume that the first eigenfunctions are normalized we derive from (8) that

$$t \leqq \left(\int_Q \Phi\varphi_u dx\right)^{-1}\left[C\,|\,Q\,|^{1/2} + \left(\int_Q h^2 dx\right)^{1/2}\right].$$

Here $|\,Q\,|$ denotes the n-dimensional Lebesgue measure of Q. By Lemma 1

$$t \leqq \left(\inf_Q \Phi \cdot \inf_{v \in L^2(Q)} \int_Q \varphi_v dx\right)^{-1}\left[C\,|\,Q\,|^{1/2} + \left(\int_Q h^2 dx\right)^{1/2}\right],$$

and this completes the proof.

Lemma 3. *For each fixed t the problem* (1_t), (2) *has a subsolution.*

Proof. Let us consider the Dirichlet problem

$$Lu = \delta_1 u - C + t\Phi + h \qquad \text{in } Q,$$

$$u(x) = 0 \qquad \text{on } \partial Q,$$

where δ_1 and C are constants appearing in the inequality (4). Since $\delta_1 < \kappa_1$, by the Schauder fixed point theorem this problem admits at least one solution $U \in \dot{W}^{1,2}(Q)$. Taking C large, to ensure that $t\Phi + h < C$ on Q, by the maximum principle we then obtain $U \leqq 0$ on Q. The fact that U is a subsolution of (1_t), (2) follows from the estimate (4).

Lemma 4. *There exists t such that the problem* (1_t), (2) *has a supersolution.*

Proof. We follow here the argument from the proof of Lemma 6 in [16]. Let $Q_1 \subset \bar{Q}_1 \subset Q_2 \subset \bar{Q}_2 \subset Q$ with $|\,Q - Q_1\,| = \delta$ to be chosen later. For a fixed $N > 0$ we define

$$m = \sup\{f(s) + h(x);\, x \in Q,\, 0 \leqq s \leqq N\}.$$

Let $H \in C(\bar{Q})$ satisfy $H(x) = |\,m\,|$ on $Q - Q_2$, $H(x) = 0$ on Q_1 and $0 \leqq H \leqq |\,m\,|$ on Q. We denote by V a solution to the Dirichlet problem

$$Lu = H \quad \text{in } Q, \qquad u = 0 \quad \text{on } \partial Q.$$

The existence of a solution $V \in \dot{W}^{1,2}(Q)$ easily follows from the Schauder fixed point theorem. Since $H \geqq 0$ on Q, it follows from the maximum principle that $V \geqq 0$ on Q. By standard estimates (see [17], Theorem 8.16 p. 191) we have

$$V(x) \leqq C_1 \| H \|_{L^s} \leqq C_1 |m| |Q - Q_1|^{1/s} = C_1 |m| \delta^{1/s}$$

where $s > n$ and $C_1 > 0$ is a constant independent of V. We now choose δ so that $C_1 |m| \delta^{1/s} \leqq N$. It is obvious that $|m| + t\Phi \leqq H$ on Q for t sufficiently large and negative. Thus

$$LV = H \geqq m + t\Phi \geqq f(x, V) + t\Phi + h \qquad \text{on } Q,$$

that is, V is a supersolution.

3. Existence of at least one solution

We now are in a position to establish the following existence result.

THEOREM 1. *There exists t^* such that the problem* (1_t), (2) *has at least one solution for $t < t^*$ and no solution for $t > t^*$.*

PROOF. By Lemma 4 for t large and negative there exists a supersolution $V \in \dot{W}^{1,2}(Q)$ of (1_t), (2) which is non-negative on Q. Lemma 3 implies the existence of a subsolution $U \in \dot{W}^{1,2}(Q)$ of (1_t), (2). It follows, from the proof of Lemma 3 that $U \leqq 0$ on Q, hence $U \leqq V$ on Q. Applying the result of Deuel and Hess [15] we conclude the existence of a solution $u \in \dot{W}^{1,2}(Q)$ of the problem (1_t), (2) such that $U \leqq u \leqq V$ on Q. Now we shall show that if there exists a solution to the problem (1_t), (2) for $t = \bar{t}$, then there exists also a solution to this problem for each $t < \bar{t}$. Indeed, let $t < \bar{t}$ and $\bar{u} \in \dot{W}^{1,2}(Q)$ be a solution of the problem $(1_{\bar{t}})$, (2). Thus

$$L\bar{u} = f(\bar{u}) + \bar{t}\Phi + h \geqq f(\bar{u}) + t\Phi + h \qquad \text{in } Q,$$

and consequently $\bar{u}$ is a supersolution of the problem (1_t), (2). It follows from the estimate (4) that

$$L\bar{u} > \delta_1 \bar{u} - C + t\Phi + h \qquad \text{in } Q.$$

Since $\delta_1 < \kappa_1$, the Dirichlet problem

$$L^{\bar{u}} v = \delta_1 v - C + t\Phi + h \quad \text{in } Q, \qquad v = 0 \quad \text{on } \partial Q$$

has at least one solution $v \in \dot{W}^{1,2}(Q)$. By virtue of Theorem 8.16 in [17], $v \in L^\infty(Q)$. We now notice that

$$L^a(\bar{u}-v) \geqq \delta_1(\bar{u}-v) \qquad \text{in } Q,$$

$$\bar{u}-v=0 \qquad \text{on } \partial Q,$$

hence by the maximum principle $\bar{u} \geqq v$ on Q. Consequently there exists a constant $M>0$ such that $\bar{u} \geqq -M$ on Q. Finally, let us consider the Dirichlet problem

$$Lu=\delta_1 u - C + t\Phi + h \qquad \text{in } Q,$$

$$u=-M \qquad \text{on } \partial Q.$$

By the Schauder fixed point theorem this problem has at least one solution $V \in W^{1,2}(Q)$. Since we may assume that $C>t\Phi+h$ on Q, the maximum principle yields $V \leqq -M$ on Q. It then follows from (4) that

$$LV \leqq f(V)+t\Phi+h \qquad \text{in } Q,$$

that is, V is a subsolution of the problem (1_t), (2) such that $V \leqq \bar{u}$ on Q. Applying again the result of Deuel and Hess [15] we deduce the existence of a solution $u \in \dot{W}^{1,2}(Q)$ of the problem (1_t), (2) such that $V \leqq u \leqq \bar{u}$ on Q. To complete the proof we set

(11) $\quad t^* = \sup\{t;$ the problem (1_t), (2) admits a solution in $\dot{W}^{1,2}(Q)\}$.

To establish the existence of a solution of (1_{t^*}), (2) we need the concept of the G-convergence (see Spagnolo [24]).

Let $H^{-1}(Q)=(\dot{W}^{1,2}(Q))^*$ and set

$$M(\alpha,\beta,Q)=\Big\{A \in L(\dot{W}^{1,2}(Q),H^{-1}(Q));\, A=-\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j\,\cdot)$$

$$\text{with } a_{ij}\in L^\infty(Q),\ a_{ij}=a_{ji}\ (i,j=1,\ldots,n) \text{ and}$$

$$\alpha|\xi|^2 \leqq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leqq \beta|\xi|^2 \text{ for all } \xi\in\mathbf{R}_n \text{ and a.e. } x\in Q\Big\}.$$

A sequence of operators $\{A_\delta\}$ $(\delta>0)$ in $M(\alpha,\beta,Q)$ is said to be G-convergent to $A_0 \in M(\alpha,\beta,Q)$ iff for each $f\in H^{-1}(Q)$ we have $\lim_{\delta\to 0} A_\delta^{-1}f = A_0^{-1}f$ weakly in $\dot{W}^{1,2}(Q)$.

The following result is due to Spagnolo [24]. Let $\{A_\delta\}$ be a sequence in $M(\alpha,\beta,Q)$, then there exist $A_0\in M(\alpha,\beta^1,Q)$ for some $\beta^1>0$ and a subsequence $\{A_{\delta'}\}$ of $\{A_\delta\}$ such that $A_{\delta'}$ G-converges to A_0.

It is also known that if A_δ G-converges to A_0, then the eigenvalues of A_δ

converge to eigenvalues of A_0. Also eigenfunctions of A_δ converge weakly in $\dot{W}^{1,2}(Q)$ to eigenfunctions of A_0 (see [4]).

Finally, if the sequence $\{A_\delta\}$ ($\delta \geqq 0$) in $M(\alpha, \beta, Q)$ is G-convergent to A_0 and if $\{v_\delta\}$ in $\dot{W}^{1,2}(Q)$ satisfies

(a) $\lim_{\delta\to 0} v_\delta = v$ weakly in $\dot{W}^{1,2}(Q)$,

(b) $\lim_{\delta\to 0} A_\delta v_\delta = f$ in $H^{-1}(Q)$,

then $A_0 v = f$ (see Spagnolo [24]).

THEOREM 2. *The problem* (1_{t^*}), (2) *has at least one solution in* $\dot{W}^{1,2}(Q)$.

PROOF. By Lemma 1 t^*, defined by (11), is finite. Let $t_k < t^*$ and $t_k \to t^*$ as $k \to \infty$. By Theorem 1 for every k there exists at least one solution u_k in $\dot{W}^{1,2}(Q)$. We now show that the sequence u_k is bounded in $\dot{W}^{1,2}(Q)$. In the contrary case we may assume that $\| u_k \|_{W^{1,2}} \to \infty$ as $k \to \infty$. It is easy to deduce from (A) and (B) that

$$\int_Q |Du_k(x)|^2 dx \leqq C_1 \int_Q u_k(x)^2 dx + C_2 \tag{12}$$

for some constants $C_1 > 0$ and $C_2 > 0$. Therefore we may assume that $\lim_{k\to\infty} \| u_k \|_{L^2} = \infty$. The inequality (12) also shows that $v_k = u_k \| u_k \|_{L^2}^{-1}$ is bounded in $\dot{W}^{1,2}(Q)$. Consequently we may assume that there exists a function $v \in \dot{W}^{1,2}(Q)$ such that $\lim_{k\to\infty} v_k = v$ weakly in $\dot{W}^{1,2}(Q)$, strongly in $L^2(Q)$ and a.e. on Q. We may also assume that the sequence of operators L^{u_k} is G-convergent to an operator $B \in M(\alpha, \beta^1, Q)$ for some $\beta^1 > 0$. Since

$$\lim_{k\to\infty} \int_Q \left[\frac{f(u_k)}{\| u_k \|_{L^2}} + t_k \frac{\Phi}{\| u_k \|_{L^2}} + \frac{h}{\| u_k \|_{L^2}} \right] \psi dx$$
$$= \int_Q f'(\infty) v^+ \Psi dx - \int_Q f'(-\infty) v^- \psi dx$$

for each $\psi \in \dot{W}^{1,2}(Q)$, we derive from the remark preceding this theorem that v satisfies the equation

$$Bv = f'(\infty)v^+ - f'(-\infty)v^- \qquad \text{in } Q \tag{13}$$

and moreover $\| v \|_{L^2} = 1$. Since the eigenvalues of L^{u_k} converge to eigenvalues of B, it is clear that the first eigenvalue ν_1 of B satisfies the inequality $\kappa_1 \leqq \nu_1 \leqq \kappa_2$. It then follows from (B) that $f'(-\infty) < \nu_1 < f'(\infty)$. Hence it is easy to see that the only solution in $\dot{W}^{1,2}(Q)$ of (13) is a trivial solution and we obtain a contradiction. Consequently the sequence $\{u_k\}$ is bounded in

$\dot{W}^{1,2}(Q)$. It is now routine to show that a subsequence of $\{u_k\}$ converges weakly in $\dot{W}^{1,2}(Q)$, strongly in $L^2(Q)$ and a.e. on Q to a solution $u \in \dot{W}^{1,2}(Q)$ of the problem (1_{t^*}), (2).

4. Existence of multiple solutions

Throughout this section we assume (A) and (B). To establish the existence of multiple solutions we use the degree theory for operators of class $(S)_+$ (see Berkovits [7], Berkovits and Mustonen [6] or Browder [9]).

We recall that a mapping f of a subset G of a Banach space X into its dual X^* is said to be of class $(S)_+$ if for any sequence $\{x_j\}$ in G which converges weakly to x in X and for which $\overline{\lim}_{j\to\infty}\langle f(x_j), x_j - x\rangle \leqq 0$ we have $x_j \to x$.

First we observe that the left side of the equation (1_t) gives rise to a bounded and continuous operator T_t of class $(S)_+$ from $\dot{W}^{1,2}(Q)$ into its dual space $H^{-1}(Q)$, defined by the formula

$$\langle T_t(u), v\rangle = \int_Q \left[\sum_{i,j=1}^{n} a_{ij}(x,u)D_iuD_jv - f(u)v - t\Phi v - hv\right] dx$$

for all u and v in $\dot{W}^{1,2}(Q)$.

We commence with an *a priori* bound for solutions of the problem (1_0), (2), where the equation (1_0) has the form $Lu = f(u) + h$.

LEMMA 5. *Let H be a bounded subset of $L^\infty(Q)$. Then there exists a constant $C(H) > 0$ such that for any solution $u \in \dot{W}^{1,2}(Q)$ of the problem* (1_0), (2) *with $h \in H$ we have*

$$\| u \|_{W^{1,2}(Q)} \leqq C(H).$$

PROOF. In the contrary case there exists a sequence $\{h_m\}$ in H and the corresponding sequence of solutions $\{u_m\}$ in $\dot{W}^{1,2}(Q)$ of the problem (1_0), (2) such that $\lim_{m\to\infty} \| u_m \|_{W^{1,2}} = \infty$. As in the proof of Theorem 2 we show also that $\lim_{m\to\infty} \| u_m \|_{L^2} = \infty$ and the sequence $v_m = u_m \| u_m \|_{L^2}^{-1}$ is bounded in $\dot{W}^{1,2}(Q)$. Therefore we may assume that v_m converges to a function $v \in \dot{W}^{1,2}(Q)$ weakly in $\dot{W}^{1,2}(Q)$, strongly in $L^2(Q)$ and a.e. on Q. Using the concept of the G-convergence we show that v satisfies the equation

$$Bv = f'(\infty)v^+ - f'(-\infty)v^-,$$

where $B \in M(\alpha, \beta^1, Q)$ for some $\beta^1 > 0$. This fact gives a contradiction (for details see the proof of Theorem 2).

LEMMA 6. *For each $t \leqq t^*$ there exists a constant $R(t) > 0$ such that the degree of T_t satisfies*

$$d(T_t, B_R(0), 0) = 0$$

for each $R \geqq R(t)$, where $B_R(0)$ is a ball centred at 0 of radius R in $\dot{W}^{1,2}(Q)$.

This result is a consequence of Lemmas 5 and 2 and the standard homotopy argument.

To prove that the degree is equal to 1 in an open and bounded set in $\dot{W}^{1,2}(Q)$ we use some ideas of McKenna and Walter [20].

We recall that a function $\delta : (0, \infty) \to (0, \infty)$ is called a modulus of continuity if it is increasing on $(0, \infty)$ and satisfies $\lim_{s \to 0+} \delta(s) = 0$.

The following result can be found in McKenna and Walter [20] (Lemma 1).

Let K be a compact set in $L^2(Q)$ and let Φ be positive a.e. on Q. Then there exists a modulus of continuity δ, depending only on K and Φ, such that

$$\| (|\varphi| - \Phi/\xi)_+ \|_{L^2} \leqq \delta(\xi) \qquad \text{for } \xi > 0 \quad \text{and} \quad \varphi \in K.$$

Since $f'(-\infty) < \kappa_1$, for each $u \in L^2(Q)$ the Dirichlet problem

$$L^u z - f'(-\infty) z = g(u) + h \quad \text{in } Q, \qquad z = 0 \quad \text{on } \partial Q$$

has a unique solution $z \in \dot{W}^{1,2}(Q)$ such that

$$\| Dz \|_{L^2(Q)} \leqq B \, \| g(u) + h \|_{L^2(Q)}, \tag{14}$$

for some constant $B > 0$ independent of u and for any g with $g' \in L^\infty(\mathbf{R})$.

It is also well known that there exists a constant $P > 0$ such that

$$\| v \|_{L^2(Q)} \leqq P \, \| Dv \|_{L^2(Q)} \tag{15}$$

for each $v \in \dot{W}^{1,2}(Q)$.

For each $v \in L^2(Q)$ let $\Phi_v \in \dot{W}^{1,2}(Q)$ be a solution of the Dirichlet problem

$$L^v u = \Phi \quad \text{in } Q \qquad \text{and} \qquad u(x) = 0 \quad \text{on } \partial Q.$$

Let us set $\Phi_1(x) = \inf_{v \in L^2(Q)} \Phi_v(x)$; obviously $\Phi_1 \in L^\infty(Q)$, moreover it follows from Theorem 8.18 in [17] that $\Phi_1(x) > 0$ on Q. For each $\varepsilon > 0$ and $t \in \mathbf{R}$ we set

$$O_{t,\varepsilon} = \bigcup_{v \in L^2(Q)} (t\Phi_v + t\varepsilon B),$$

where $B = B(0)$ is an open unit ball in $\dot{W}^{1,2}(Q)$. It follows from the assumption (A) that

$$\sup_{v \in L^2(Q)} \| D\Phi_v \|_{L^2} \leqq C \| \Phi \|_{L^2}$$

for some constant $C > 0$. Consequently the $O_{t,\varepsilon}$ is bounded for each t and ε. We now observe that since $f'(-\infty) < \kappa_1$ the operator

$$(L^v - f'(-\infty))^{-1} : L^2(Q) \to L^2(Q)$$

exists and is compact. Let

$$K = \bigcup_{v \in L^2(Q)} \overline{\{(L^v - f'(-\infty))^{-1}(\bar{B})\}}.$$

It follows from the compactness of the embedding of $\dot{W}^{1,2}(Q)$ into $L^2(Q)$ that K is a compact subset of $L^2(Q)$. Finally let δ be the modulus of continuity of K with respect to Φ_1.

LEMMA 7. *There exists $\varepsilon > 0$ and t_0 such that $d(T_t, O_{t,\varepsilon}, 0) = 1$ for $t \leqq t_0$.*

PROOF. We follow here the argument of Theorem 1 in [20]. First we show that one can choose t_0 and $\varepsilon > 0$ such that equation $T_t = 0$ has no solution on the boundary of $O_{t,\varepsilon}$ for $t \leqq t_0$.

The equation (1_t) can be written in the form

$$(1_s) \qquad Lu = f(u) + s(-\Phi) + h \qquad \text{in } Q,$$

where $s = -t$. Let $A = \sup_{\mathbf{R}} |f'(t) - f'(\infty)|$. We now choose $\varepsilon > 0$ and q so that

$$(16) \qquad \delta(\varepsilon(AP + B_1)) \leqq \frac{B_1}{4(AP + B_1)A}$$

and

$$(17) \qquad q(\| \Phi_w \|_{L^2} + \varepsilon P) \leqq \varepsilon B_1/4 \qquad \text{for all } w \in L^2(Q),$$

where P and $B_1 = B^{-1}$ are constants appearing in the inequalities (14) and (15), respectively. With the change of the parameter $s = -t$ we set

$$\tilde{Q}_{s,\varepsilon} = \bigcup_{v \in L^2(Q)} (s\bar{\Phi}_v - s\varepsilon B),$$

where $\bar{\Phi}_v = -\Phi_v$. As in [20] we write (1_s) in the form

$$(18) \qquad Lu - f'(-\infty)u = g(u) + s(-\Phi) + h,$$

with $g(u) = f(u) - f'(-\infty)u$. The function g can be written in the form

$$g(u) = g_0(u) + g_1(u) + g_2(u),$$

where $g_0(0) = 0$ and g_0' is bounded by q, g_1 is bounded, $g_2(u) = 0$ for $u \leqq 0$ and $|g_2'(u)| \leqq A$. Let u be a solution of (1_s) in $\dot{W}^{1,2}(Q)$ belonging to $\partial O_{s,\varepsilon}$, that is $u \in s\bar{\Phi}_v - s\varepsilon\partial B$ for some $v \in L^2(Q)$ and $u \in s\bar{\Phi}_v - s\varepsilon\bar{B}$ for all $w \in L^2(Q)$ and $w \neq v$. We show that this leads to a contradiction for some $\varepsilon > 0$ and s sufficiently large. It is clear that

$$\| g_0(u) \|_{L^2} \leqq qs(\| \Phi_v \|_{L^2} + \varepsilon P) \leqq s\varepsilon B_1/4,$$

$$\| h \|_{L^2} + \| g_1(u) \| \leqq s\varepsilon B_1/4 \qquad \text{for } s \geqq s_0,$$

for sufficiently large s_0 and

$$\| g_2(u) \|_{L^2} \leqq A \| u_+ \|_{L^2}.$$

Now for some $w \in \bar{B}$ we have

$$\| u_+ \|_{L^2} \leqq \| (s\bar{\Phi}_v + s\varepsilon w)_+ \|_{L^2} \leqq \| s\varepsilon w \|_{L^2} \leqq Ps\varepsilon,$$

since $\bar{\Phi}_v \leqq 0$ on Q. Consequently

$$\| h \|_{L^2} + \| g(u) \|_{L^2} \leqq B_1\varepsilon s + APs\varepsilon = s\varepsilon(B_1 + AP). \tag{19}$$

Now equation (18) can be written as

$$u = s\bar{\Phi}_u + (L^u - f'(-\infty))^{-1}(g(u) + h) \tag{20}$$

and by estimate (19) we obtain

$$u = s\bar{\Phi}_u + s\varepsilon(AP + B_1)\psi,$$

for some $\psi \in K$. Therefore we see that

$$\| u_+ \|_{L^2} \leqq s \| (\bar{\Phi}_u + \varepsilon(AP + B_1)\psi)_+ \|_{L^2} \leqq s \| (-\Phi_1 + \varepsilon(AP + B_1)\psi)_+ \|_{L^2}$$
$$\leqq s\varepsilon(AP + B_1)\delta(\varepsilon(AP + B_1)) \leqq B_1 s\varepsilon/4A.$$

This estimate implies that

$$\| g_2(u) \|_{L^2} \leqq A \| u_+ \|_{L^2} \leqq s\varepsilon B_1/4$$

and consequently

$$\| g(u) \|_{L^2} + \| h \|_{L^2} \leqq 3s\varepsilon B_1/4.$$

Combining the representation (20), estimate (14) and the last inequality we obtain

$$\| D(u - s\bar{\Phi}_u) \|_{L^2} = \| D(L^u - f'(-\infty))^{-1}(g(u) + h) \|_{L^2}$$
$$\leqq B \| g(u) + h \|_{L^2} \leqq \tfrac{3}{4}s\varepsilon,$$

which means that $u \in s\bar{\Phi}_u + \frac{3}{4}s\varepsilon\bar{B}$. This contradicts the fact that $u \in \partial O_{s,\varepsilon}$. This estimation remains true for the equation

$$L^{\lambda u}u - f'(-\infty)u = s\Phi + \lambda(h + g(u)), \qquad 0 \leqq \lambda \leqq 1.$$

To complete the proof we observe that the degree of the operator associated with the Dirichlet problem

$$L^0 u = s\Phi + f'(-\infty)u \quad \text{in } Q, \qquad u(x) = 0 \quad \text{on } \partial Q$$

is equal to 1 and the result follows applying the homotopy argument.

Now by combining Lemmas 6 and 7 with the excision property of the degree we easily obtain the following result.

THEOREM 3. *There exists t_0 such that for every $t \leqq t_0$ the Dirichlet problem* (1_t), (2) *has at least two solutions.*

It is obvious that the numbers t^* and t_0 from Theorems 1 and 3 satisfy the inequality $t_0 \leqq t^*$.

5. The Dirichlet problem in a weighted Sobolev space; nonresonant case

In this section we additionally assume that

(C) The coefficients a_{ij} $(i, j = 1, \ldots, n)$ are continuous on $\bar{Q} \times \mathbf{R}$ and for each $u \in \mathbf{R}$, $a_{ij}(\cdot, u) \in C^1(\bar{Q})$ and moreover there exist functions $A_{ij} \in C^1(\bar{Q})$ such that

$$\lim_{|u| \to \infty} a_{ij}(x, u) = A_{ij}(x) \quad \text{and} \quad \lim_{|u| \to \infty} D_x a_{ij}(x, u) = D_x A_{ij}(x)$$

$(i, j = 1, \ldots, n)$ uniformly in $x \in \bar{Q}$.
We assume that the boundary ∂Q is of class C^2.

With these limit coefficients we associate the elliptic operator

$$A = -\sum_{i,j=1}^{n} D_i(A_{ij}(x) D_j \cdot).$$

Let μ_1 denote the first eigenvalue of the operator A. We always have $\kappa_1 \leqq \mu_1$.

One can give examples of elliptic quasilinear operators for which both cases $\kappa_1 = \mu_1$ or $\kappa_1 < \mu_1$ occur.

Under this additional hypothesis we consider the Dirichlet problem

$$Lu = f(x, u) \quad \text{in } Q, \tag{21}$$

$$u(x) = \varphi(x) \quad \text{on } \partial Q, \tag{22}$$

where $\varphi \in L^2(\partial Q)$. In paper [13] we considered the Dirichlet problem (21) and (22) in both situations $\kappa_1 = \mu_1$ and $\kappa_1 < \mu_1$ and we proved that the problem (21), (22) admits a solution in a weighted Sobolov space provided $f(x, u) = \mu u$ with $\mu < \kappa_1 = \mu_1$, where μ is constant. A similar existence result was proved in the case where $\mu \leqq \kappa_1$ with $\kappa_1 < \mu_1$. The objective of this section is to improve these existence results and show that the problem (21), (22) is solvable if $\mu < \mu_1$, regardless of the relation between μ_1 and κ_1.

We impose the following condition on the nonlinearity f:

(D) The nonlinearity f satisfies the Carathéodory condition and

$$|f(x, t)| \leqq a|t| + b \quad \text{on } Q \times \mathbf{R},$$

where $a > 0$ and $b > 0$ are constants.

Since the boundary datum $\varphi \in L^2(\partial Q)$ and not every function from this space is a trace of an element from $W^{1,2}(Q)$ we cannot expect a solution of (21), (22) to belong to $W^{1,2}(Q)$. It follows from [10] and [11] (see also [22]) that the suitable Sobolev space in our situation is

$$\tilde{W}^{1,2}(Q) = \left\{u;\, u \in W^{1,2}_{\mathrm{loc}}(Q) \text{ and } \int_Q [|Du(x)|^2 r(x) + u(x)^2]dx < \infty\right\}$$

equipped with the norm

$$\| u \|^2_{\tilde{W}^{1,2}(Q)} = \int_Q [|Du(x)|^2 r(x) + u(x)^2]dx,$$

where $r(x) = \operatorname{dist}(x, \partial Q)$ for $x \in Q$.

To explain the meaning of the boundary condition (22) we need some terminology. It follows from the regularity of the boundary ∂Q that there exists a number $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$ the domain

$$Q_\delta = Q \cap \left\{x;\, \min_{y \in \partial Q} |x - y| > \delta\right\}$$

with the boundary ∂Q_δ possesses the following property: to each $x_0 \in \partial Q$ there is a unique point $x_\delta(x_0) \in \partial Q_\delta$ such that $x_\delta(x_0) = x_0 - \delta\nu(x_0)$, where $\nu(x_0)$ is the outward normal ∂Q at x_0. The above relation gives a one-to-one mapping, of class C^1, of ∂Q on ∂Q_δ.

As in [10] (see also [11] and [22]) we adopt the following approach to the Dirichlet problem (21), (22).

Let $\varphi \in L^2(\partial Q)$. A weak solution $u \in W^{1,2}_{\mathrm{loc}}(Q)$ of (21) is a solution of the Dirichlet problem with the boundary condition (22) if

$$\lim_{\delta \to 0} \int_{\partial Q} [u(x_\delta(x)) - \varphi(x)]^2 dS_x = 0. \tag{23}$$

It follows from [11] that any solution $u \in W^{1,2}_{\mathrm{loc}}(Q)$ of (21), (22), with the boundary condition in the sense of the L^2-convergence, must belong to $\tilde{W}^{1,2}(Q)$.

Since $\partial Q \in C^2$, the distance function $r(x)$ belongs to $C^2(\bar{Q} - Q_{\delta_0})$, if δ_0 is sufficiently small. We denote by $\rho(x)$ the extension of the function $r(x)$ into $\bar{Q}$ satisfying the following properties: $\rho(x) = r(x)$ on $\bar{Q} - Q_\delta$, $\rho \in C^2(\bar{Q})$ and $\rho(x) > 0$ on Q.

Theorem 4. *Let $\kappa_1 < \mu_1$. If*

$$\lim_{|t| \to \infty} \frac{f(x, t)}{t} = \mu < \mu_1$$

uniformly in $x \in Q$, then the Dirichlet problem (21), (22) *has at least one solution $u \in \tilde{W}^{1,2}(Q)$.*

Proof. For $\varepsilon > 0$ we set

$$f_\varepsilon(x, u) = \frac{f(x, u)}{1 + \varepsilon |f(x, u)|} \qquad \text{for } (x, u) \in Q \times \mathbf{R}$$

and consider for each $\varepsilon > 0$ the Dirichlet problem

$$Lu = f_\varepsilon(x, u) \qquad \text{in } Q, \tag{21ε}$$

with the boundary condition (22). Since $|f_\varepsilon(x, u)| \leqq 1/\varepsilon$ on $Q \times \mathbf{R}$, it follows from [12] that for each $\varepsilon > 0$ there exists a solution $u_\varepsilon \in \tilde{W}^{1,2}(Q)$ of the problem (21ε), (22).

The family of solutions $\{u_\varepsilon, \varepsilon > 0\}$ must be bounded in $L^2(Q)$. In the contrary case we may assume that $\| u_{\varepsilon_m} \|_{L^2} \to \infty$ for some sequence $\varepsilon_m \to 0$. Let us set $u_m = u_{\varepsilon_m}$ and $v_m = u_m \| u_m \|_{L^2}^{-1}$ and let

$$\psi(x) = \begin{bmatrix} u_m(x)(\rho(x) - \delta) & \text{on } Q_\delta \\ 0 & \text{on } Q - Q_\delta \end{bmatrix}$$

for $0 \leqq \delta \leqq \delta_0$. Taking ψ as a test function and integrating by parts we get

$$\begin{aligned} &\int_{Q_\delta} \sum_{i,j=1}^{n} a_{ij}(x, u_m) D_i u_m D_j u_m (\rho - \delta) dx \\ &\quad - \frac{1}{2} \int_{\partial Q_\delta} \sum_{o,j=1}^{n} \int_0^{u_m^2} a_{ij}(x, s) ds D_i \rho D_j \rho dS_x \\ (24) &\quad - \frac{1}{2} \int_{Q_\delta} \sum_{i,j=1}^{n} \int_0^{u_m^2} a_{ij}(x, s) ds D_{ij} \rho dx - \frac{1}{2} \int_{Q_\delta} \sum_{i,j=1}^{n} \int_0^{u_m^2} D_i a_{ij}(x, s) ds D_j \rho dx \\ &\leqq \int_{Q_\delta} |f(x, u_m)| \, |u_m| (\rho - \delta) dx. \end{aligned}$$

Letting $\delta \to 0$ and applying the ellipticity condition and the Young inequality we obtain

$$\int_Q |Du_m|^2 \rho dx \leqq C_1 \int_Q u_m^2 dx + C_2 \int_Q \varphi^2 dS_x + C_3 \tag{25}$$

for some positive constants C_1, C_2 and C_3. This inequality implies that the sequence v_m is bounded in $\tilde{W}^{1,2}(Q)$. Consequently, by Theorem 4.11 in [21], we may assume that $\lim_{m \to \infty} v_m = v$ weakly in $\tilde{W}^{1,2}(Q)$, strongly in $L^2(Q)$ and a.e. on Q. Since $\| v_m \|_{L^2} = 1$ for each m, $v \not\equiv 0$ on Q. We now observe that

$$\frac{1}{1 + \varepsilon_m |f(x, u_m)|} \leqq 1 \qquad \text{on } Q \quad \text{for all } m$$

and therefore we may assume that

$$\frac{1}{1 + \varepsilon_m |f(x, u_m)|} \quad \text{converges weakly in } L^2(Q) \text{ to some function } a \in L^2(Q).$$

It is also clear that

$$\int_Q a(x) w(x)^2 dx \leqq \int_Q w(x)^2 dx \tag{26}$$

for each $w \in L^2(Q)$. Let ψ be a function in $W^{1,2}(Q)$ with compact support in Q, thus

$$(27)\qquad \int_Q \sum_{i,j=1}^n a_{ij}(x,u_m)D_i v_m D_j\psi dx = \int_Q f_{\varepsilon_m}(x,u_m)\,\|u_m\|_{L^2}^{-1}\psi(x)dx.$$

Writing the right hand side of (2) as

$$\int_Q \frac{f_{\varepsilon_m}(x,u_m)}{\|u_m\|_{L^2}}\psi(x)dx = \left[\int_Q \frac{f(x,u_m)}{\|u_m\|_{L^2}} - \mu v(x)\right]\frac{\psi(x)}{1+\varepsilon_m|f(x,u_m)|}dx$$

$$+\mu\int_Q v(x)\frac{\psi(x)}{1+\varepsilon_m|f(x,u_m)|}dx$$

and using the assumption that

$$\lim_{|t|\to\infty}\frac{f(x,t)}{t} = \mu \qquad \text{uniformly in } Q,$$

we can easily show that

$$(28)\qquad \lim_{m\to\infty}\int_Q \frac{f_{\varepsilon_m}(x,u_m)}{\|u_m\|_{L^2}}\psi(x)dx = \mu\int_Q a(x)v(x)\psi(x)dx.$$

On the other hand

$$\int_Q \sum_{i,j=1}^n a_{ij}(x,u_m)D_i v_m D_j\psi dx = \|u_m\|_{L^2}^{-1}\int_Q \sum_{i,j=1}^n D_i\int_0^{u_m} a_{ij}(x,s)dsD_j\psi dx$$

$$-\|u_m\|_{L^2}^{-1}\int_Q \sum_{i,j=1}^n \int_0^{u_m} D_i a_{ij}(x,s)dsD_j\psi dx$$

$$= -\|u_m\|_{L^2}^{-1}\int_Q \sum_{i,j=1}^n \int_0^{u_m} a_{ij}(x,s)dsD_{ij}\psi dx$$

$$-\|u_m\|_{L^2}^{-1}\int_Q \sum_{i,j=1}^n \int_0^{u_m} D_i a_{ij}(x,s)dsD_j\psi dx$$

and repeating the argument from [12] we deduce that

$$\lim_{m\to\infty}\int_Q \sum_{i,j=1}^n a_{ij}(x,u_m)D_iu_mD_j\psi dx = -\int_Q \sum_{i,j=1}^n A_{ij}D_{ij}\psi v dx - \int_Q \sum_{i,j=1}^n D_iA_{ij}vD_j\psi dx.$$

Combining the last relation with (27) and (28) we get

$$\int_Q \sum_{i,j=1}^n A_{ij}(x)D_ivD_j\psi dx = \mu\int_Q a(x)v\psi dx \tag{29}$$

for each $\psi\in W^{1,2}(Q)$ with compact support in Q. It follows from [10] that v has a trace $\xi\in L^2(\partial Q)$. Replacing in (29) ψ by $\psi\cdot\rho$ with $\psi\in W^{1,2}(Q)$ and integrating by parts we get

$$\begin{aligned}\int_Q \sum_{i,j=1}^n A_{ij}D_ivD_j\psi\rho dx - \int_{\partial Q}\sum_{i,j=1}^n A_{ij}\xi\psi D_i\rho D_j\rho dS_x \\ -\int_Q \sum_{i,j=1}^n D_i(A_{ij}D_j\rho v)\psi dx = \mu\int_Q a(x)v\psi dx.\end{aligned} \tag{30}$$

Similarly taking $\rho\psi$ as a test function, with $\psi\in W^{1,2}(Q)$, we get

$$\begin{aligned}&-\|u_m\|_{L^2}^{-1}\int_Q \sum_{i,j=1}^n \int_0^{u_m} D_ia_{ij}(x,s)dsD_j\psi\rho dx \\ &-\|u_m\|_{L^2}^{-1}\int_Q \sum_{i,j=1}^n \int_0^{u_m} a_{ij}(x,s)dsD_i(D_j\psi\rho)dx \\ &-\|u_m\|_{L^2}^{-1}\int_Q \sum_{i,j=1}^n \int_0^{u_m} D_ia_{ij}(x,s)ds\psi D_j\rho dx \\ &-\|u_m\|_{L^2}^{-1}\int_Q \sum_{i,j=1}^n \int_0^{u_m} a_{ij}(x,s)dsD_i(\psi D_j\rho)ds \\ &-\|u_m\|_{L^2}^{-1}\int_{\partial Q} \sum_{i,j=1}^n \int_0^{u_m} a_{ij}(x,s)dsD_i\rho dS_x \\ &= \|u_m\|_{L^2}^{-1}\int_Q f_{\varepsilon_m}(x,u_m)\psi dx.\end{aligned}$$

Letting $m\to\infty$ we derive from the last equation that

$$-\int_Q \sum_{i,j=1}^n D_iA_{ij}(x)vD_j\psi\rho dx - \int_Q \sum_{i,j=1}^n A_{ij}(x)vD_i(D_j\psi_\rho)dx$$

$$(31)\qquad -\int_Q \sum_{i,j=1}^n D_iA_{ij}(x)v\psi D_j\rho\, dx - \int_Q \sum_{i,j=1}^n A_{ij}(x)vD_i(\psi D_j\rho)dx$$

$$=\mu\int_Q a(x)v\psi dx$$

for each $\psi\in W^{1,2}(Q)$. Comparing (30) and (31) we get

$$\int_{\partial Q}\psi \sum_{i,j=1}^n A_{ij}(x)D_i\rho D_j\rho\xi dS_x = 0$$

for each $\psi\in W^{1,2}(Q)$ and consequently $\xi=0$ a.e. on ∂Q. Taking v as a test function in (29) we obtain from the variational characterization of the first eigenvalue μ_1 that

$$\begin{aligned}\mu_1\int_Q v(x)^2dx &\leqq \int_Q \sum_{i,j=1}^n A_{ij}(x)D_ivD_jvdx\\ &=\mu\int_Q a(x)v^2dx\\ &\leqq \mu\int_Q v^2dx,\end{aligned}$$

which is impossible since $\mu_1>\mu$ and $v\equiv 0$ on Q. Therefore $\{u_m\}$ is bounded in $L^2(Q)$ and by virtue of the estimate (25) the sequence $\{u_m\}$ is bounded in $\tilde{W}^{1,2}(Q)$. Now it is obvious that with the aid of Theorem 4.11 in [21] we can show that a suitable subsequence of u_m converges to solution of (21), (22) weakly in $W^{1,2}(Q)$, strongly in $L^2(Q)$ and a.e. on Q (for details see [11] and [12]).

If we assume the existence of one-sided limits in (C) we can only claim the existence of solutions for nonnegative or nonpositive boundary data. Namely, let us assume

$$(\mathrm{C}_+)\qquad \lim_{t\to\infty} a_{ij}(x,t)=A_{ij}^+(x),\qquad \lim_{t\to\infty} D_xa_{ij}(x,t)=D_xA_{ij}^+(x)$$

uniformly in $\bar{Q}$, where $A_{ij}^+\in C^1(\bar{Q})$ $(i,j=1,\dots,n)$.

Let $\mu_1{}^+$ be the first eigenvalue of the operator

$$A^+=-\sum_{i,j=1}^n D_i(A_{ij}^+(x)D_j\cdot).$$

THEOREM 5. *Suppose that* (C_+) *holds and that*

$$\lim_{t\to\infty}\frac{f(x,t)}{t}=\mu<\mu_1^+$$

uniformly in $x\in Q$ *and* $f(x,u)\geqq 0$ *for* $x\in Q$ *and* $\mu\in[0,\infty)$. *Then for each* $\varphi\in L^2(\partial Q)$, *with* $\varphi\geqq 0$ *on* ∂Q, *there exists a nonnegative solution* $u\in\tilde{W}^{1,2}(Q)$.

PROOF. For each $\varepsilon>0$ we consider the Dirichlet problem

$$(21_{\varepsilon+})\qquad Lu=\frac{|f(x,u)|}{1+\varepsilon|f(x,u)|}\quad\text{in } Q,$$

with the boundary condition (22). It follows from [12] and the maximum principle that for each $\varepsilon>0$ the problem $(21_{\varepsilon+})$, (22) has a nonnegative solution $u_\varepsilon\in\tilde{W}^{1,2}(Q)$. Consequently u_ε satisfies the equation

$$Lu_\varepsilon=\frac{f(x,u_\varepsilon)}{1+\varepsilon|f(x,u_\varepsilon)|}\quad\text{in } Q.$$

Repeating the argument of Theorem 4 the result easily follows.

Finally, let us assume that

$$(C_-)\qquad \lim_{t\to-\infty}a_{ij}(x,t)=A_{ij}^-(x),\qquad \lim_{t\to\infty}D_xa_{ij}(x,t)=D_xA_{ij}^-(x)$$

uniformly in $x\in Q$, where $A_{ij}^-\in C^1(\bar{Q})$ $(i,j=1,\dots,n)$.

Denoting by μ_1^- the first eigenvalue of the elliptic operator

$$A^-=-\sum_{i,j=1}^{n}D_i(A_{ij}^-(x)D_j\cdot)$$

we can state the following existence result.

THEOREM 6. *Suppose* (C_-) *holds and that*

$$\lim_{t\to-\infty}\frac{f(x,t)}{t}=\mu<\mu_1^-$$

uniformly in $x\in Q$ *and that* $f(x,u)\leqq 0$ *for* $x\in Q$ *and* $u\in[0,\infty)$. *Then for each* $\varphi\in L^2(\partial Q)$ *with* $\varphi\leqq 0$ *on* ∂Q, *there exists a nonpositive solution in* $\tilde{W}^{1,2}(Q)$ *of the problem* (21), (22).

6. The Dirichlet problem with the nonlinearity crossing the first eigenvalue of the limit operator

The aim of this section is to investigate the existence of solutions of the problem (1_t), (2) under the assumption (C). We commence with a partial existence result. As in Section 5 we denote by μ_1^- the first eigenvalue of the operator A^-.

THEOREM 7. *Suppose that the coefficients* a_{ij}, $i = 1, \ldots, n$, *satisfy* (C_-) *and that* f *has a bounded derivative with* $f'(-\infty) < \mu_1^-$. *Then there exists* t_0 *such that the problem* (1_t), (2) *is solvable in* $\dot{W}^{1,2}(Q)$ *for each* $t \leqq t_0$.

PROOF. It follows from the assumptions on f, that there exists $\delta_1 > 0$ and $C > 0$ such that

$$f(s) \geqq \delta_1 s - C$$

for all $s \in \mathbf{R}$. By Theorem 5 for every t, the Dirichlet problem

$$Lu = \delta_1 u - C + t\Phi + h \qquad \text{in } Q,$$

$$u(x) = 0 \qquad \text{on } \partial Q$$

has a nonpositive solution $U \in \dot{W}^{1,2}(Q)$. Here C is chosen so that $t\Phi + h \leqq C$ on Q. It is clear that U is a subsolution of the problem (1_t), (2). Inspection of Lemma 4 shows that there exists t_0 such that the problem (1_t), (2) has nonnegative supersolution $V \in \dot{W}^{1,2}(Q)$ for each $t \leqq t_0$. The existence of a solution for each $t \leqq t_0$ follows from the result of Deuel and Hess [15].

We now establish a nonexistence result.

THEOREM 8. *Suppose that the coefficients* a_{ij}, $i, j = 1, \ldots, n$, *satisfy* (C) *and that* f *has a bounded derivative on* $\mathbf{R}$ *with* $f'(-\infty) < \mu_1 < f'(\infty)$. *Then there exists* t_1 *such that the problem* (1_t), (2) *has no solution in* $\dot{W}^{1,2}(Q)$ *for all* $t \geqq t_1$.

PROOF. Suppose that our assertion is false. Then there exist a sequence $t_m \to \infty$ as $m \to \infty$ and a sequence $\{u_m\}$ in $\dot{W}^{1,2}(Q)$ such that

$$Lu_m = f(u_m) + t_m\Phi(x) + h(x) \qquad \text{in } Q,$$

$$u_m(x) = 0 \qquad \text{on } \partial Q.$$

First we prove that the sequence $\{u_m/t_m\}$ is bounded in $L^2(Q)$. In the contrary case we may assume that

$$\lim_{m\to\infty} \frac{\| u_m \|}{t_m} L^2 = \infty.$$

This also implies that $\lim_{m\to\infty} \| u_m \|_{L^2} = \infty$. Let us set $v_m = u_m / \| u_m \|_{L^2}$. Using u_m as a test function in (3) we derive the estimate

$$\int_Q |Du_m|^2 dx \leqq C_1 \int_Q u_m^2\, dx + C_2 t_m + C_3 \tag{32}$$

for some constants $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$. This estimate yields that

$$\int_Q |Dv_m|^2 dx \leqq C_1 + \frac{t_m}{\| u_m \|_{L^2}^2} C_2 + \frac{C_3}{\| u_m \|_{L^2}^2} .$$

Since $t_m / \| u_m \|_{L^2}^2$ and $1/\| u_m \|_{L^2}$ both tend to 0 as $m \to \infty$, we conclude that the sequence v_m is bounded in $\dot{W}^{1,2}(Q)$. Therefore we may assume that $\lim_{m\to\infty} v_m = v$ weakly in $W^{1,2}(Q)$, strongly in $L^2(Q)$ and a.e. on Q. Since $\| v_m \|_{L^2} = 1$ for all m, we see that $v \equiv 0$ on Q. It is also easy to see that v satisfies the equation

$$Av = f'(\infty)v^+ - f'(-\infty)v^- \qquad \text{in } Q,$$

which has only trivial solution and we arrive at a contradiction. Consequently $w_m = u_m/t_m$ must be bounded in $L^2(Q)$ and by (32) it is also bounded in $\dot{W}^{1,2}(Q)$. Therefore we may assume that $\lim_{m\to\infty} w_m = w$ weakly in $\dot{W}^{1,2}(Q)$, strongly in $L^2(Q)$ and a.e. on Q. The limit function w is a solution of the equation

$$Aw = f'(\infty)w^+ - f'(-\infty)w^- + \Phi \qquad \text{in } Q.$$

Taking as a test function the first eigenfunction Φ_1 of A we get

$$\mu_1 \int_Q w\Phi_1 dx = f'(\infty) \int_Q w^+\Phi_1 dx - f'(-\infty) \int_Q w^-\Phi_1 dx + \int_Q \Phi\Phi_1 dx,$$

which gives that

$$(f'(-\infty) - \mu_1) \int_Q w^-\Phi_1 dx = (f'(\infty) - \mu_1) \int_Q w^+\Phi_1 dx + \int_Q \Phi\Phi_1 dx.$$

Since $f'(-\infty) - \mu_1 < 0$, $f'(\infty) - \mu_1 > 0$ and $\int_Q \Phi\Phi_1 dx > 0$, we get a contradiction.

It is clear that, under the assumptions of Theorem 8, Lemma 5 continues to hold. Consequently using the homotopy argument we show that for each t there exists a $R(t) > 0$ such that

$$d(T_t, B_R(0), 0) = 0$$

for each $R \geqq R(t)$.

We are now in a position to formulate the multiplicity result.

THEOREM 9. *Suppose that the coefficients* a_{ij}, $i, j = 1, \ldots, n$, *satisfy* (C) *and that the nonlinearity f has a bounded derivative on* $\mathbf{R}$ *with* $f'(-\infty) < \kappa_1$ *and* $f'(\infty) > \mu_1$. *Then there exists* t_1 *such that the problem* (1_t), (2) *has at least two solutions for* $t \leqq t_1$.

The inequality $f'(-\infty) < \kappa_1$ implies that Lemma 7 remains true and the result follows from the excision property of the degree of the operator T_t.

Theorem 9 is a generalization of Theorem 3. We point out here that one can give an example of a quasilinear operator for which $\kappa_2 > \mu_1$ therefore the inequality $f'(\infty) > \mu_1$ not necessarily implies the second inequality from the assumption (B).

7. The Dirichlet problem in resonance

Recently Shapiro [23] has introduced the following quantity related to the spectrum of the family of the operators $\{L^v, v \in L^2(Q)\}$.

Let ψ_m be a sequence of orthonormal eigenfunctions of the Laplace operator

$$-\Delta\psi_m = \lambda_m \psi_m \qquad \text{in } Q,$$

$$\psi_m(x) = 0 \qquad \text{on } \partial Q.$$

Let

$$\lambda_1^* = \inf\left\{\frac{(L^w u, u)}{(u, u)}, u \not\equiv 0, u = C_1\psi_1 + \cdots + C_n\psi_n, \right.$$
$$w = \zeta_1\psi_1 + \cdots + \zeta_n\psi_n \quad \text{, where } n \text{ is an arbitrary integer,}$$
$$\left. C_1, \ldots, C_n \text{ and } \zeta_1, \ldots, \zeta_n \text{ are constants}\right\}.$$

λ_1^* can be viewed as a generalized first eigenvalue of the quasilinear operator L. Under the different set of assumptions on L Shapiro [23] established some existence theorem for the Dirichlet problem in resonance. Here we briefly discuss the Dirichlet problem in resonance under the assumption that the coefficients $(a_{ij}\}$, $i, j = 1, \ldots, n$, satisfy (C) with $\mu_1 = \kappa_1$.

We commence by showing that $\kappa_1 = \lambda_1^*$. Indeed, taking

$$w = w_N = N\psi_1 + \zeta_2\psi_2 + \cdots + \zeta_n\psi_n$$

and u as in the definition of λ_1^* we have

$$\lambda_1^* \leqq \frac{(L^{w_N}u, u)}{(u, u)}.$$

Letting $N \to \infty$ we obtain

$$\lambda_1^* \leqq \frac{(Au, u)}{(u, u)}.$$

Since $\{\psi_m\}$ is complete in $\dot{W}^{1,2}(Q)$ we see that $\lambda_1^* \leqq \mu_1$. On the other hand, for each $w = C_1\psi_1 + \cdots + C_n\psi_n$ and $u = C_1\psi_1 + \cdots + C_n\psi_n$, with $u \not\equiv 0$, we have

$$\kappa_1 \leqq \lambda_1^w \leqq \frac{(L^w u, u)}{(u, u)}$$

and consequently $\kappa_1 \leqq \lambda_1^*$.

THEOREM 10. *Suppose that the nonlinearity f is a Carathéodory function on $Q \times \mathbf{R}$ and $|f(x, u)| \leqq H(x)$ on $Q \times \mathbf{R}$ for some nonnegative function H in $L^2(Q)$. Suppose further that there exist limits*

$$\lim_{u \to \infty} f(x, u) = f^+(x) \quad \textit{and} \quad \lim_{n \to \infty} f(x, u) = f^-(x)$$

a.e. *on Q. If*

(33) $$\int_Q f^+(x)\Phi_1(x)dx + \int_Q h(x)\Phi_1(x)dx < 0$$

and

(34) $$\int_Q f^-(x)\Phi_1(x)dx + \int_Q h(x)\Phi_1(x)dx > 0,$$

where Φ_1 is the first eigenfunction of A, then the Dirichlet problem

$$Lu = f(x, u) + h(x) \quad \textit{in } Q \qquad \textit{and} \qquad u(x) = 0 \quad \textit{on } \partial Q$$

admits a solution in $\dot{W}^{1,2}(Q)$.

PROOF. For each integer $m \geqq 1$ we consider the Dirichlet problem

$$Lu = (\kappa_1 - 1/m)u + f(x, u) + h(x) \quad \text{in } Q \qquad \text{and} \qquad u(x) = 0 \quad \text{on } \partial Q.$$

According to Theorem 4, for each m there exists a solution $u_m \in \dot{W}^{1,2}(Q)$. It is sufficient to show that the sequence $\{u_m\}$ is bounded in $L^2(Q)$. In the contrary case we may assume that $\lim_{m\to\infty} \| u_m \|_{L^2} = \infty$. It is easy to see that

$$\int_Q |Du_m|^2 dx \leqq C_1 \int_Q u_m^2 dx + C_2$$

for some constants $C_1 > 0$ and $C_2 > 0$. Consequently the sequence $v_m = u_m / \| u_m \|_{L^2}$ is bounded in $\dot{W}^{1,2}(Q)$.

It is now routine to show that a subsequence of v_m, which we relabel again as v_m, converges to a function v weakly in $\dot{W}^{1,2}(Q)$, strongly in $L^2(Q)$ and a.e. on Q. Moreover v satisfies the equation $Av = \kappa_1 v$. Since $\| v \|_{L^2} = 1$, therefore either $v = \Phi_1$ or $v = -\Phi_1$. It follows from the variational characterization of eigenvalues that

$$\frac{1}{m} \int_Q u_m^2 dx \leqq \int_Q f(x, u_m) u_m dx + \int_Q h(x) dx$$

and this implies that

$$0 \leqq \int_Q f(x, v_m \| u_m \|_{L^2}) v_m dx + \int_Q h v_m dx.$$

If $v = \Phi_1$ we get, letting $m \to \infty$, that

$$0 \leqq \int_Q f^+(x) \Phi_1(x) dx + \int_Q h(x) \Phi_1(x) dx,$$

which contradicts (32). Similarly, if $v = -\Phi_1$ we get

$$0 \geqq \int_Q f^-(x) \Phi_1(x) dx + \int_Q h(x) \Phi_1(x) dx$$

which gives a contradiction to (34).

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